MORE ON THE PRESSING DOWN GAME.

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Abstract. We investigate the pressing down game and its relation to the Banach Mazur game. In particular we show: Consistently relative to a supercompact, there is a nowhere precipitous normal ideal I on $\aleph_2$ such that player nonempty wins the pressing down game of length $\aleph_1$ on I even if player empty starts. For the proof, we construct a forcing notion to force the following: There is normal, nowhere precipitous ideal I on a supercompact κ such that for every I-positive A there is a normal ultrafilter containing A and extending the dual of I.

We investigate the pressing down game and its relation to the Banach Mazur game. Definitions (and some well known or obvious properties) are given in Section 1. The results are summarized in Section 2. This paper continues (and simplifies, see 2.2) the investigation of Pauna and the authors in [14].

1. Definitions

We use the following notation:

- For forcing conditions $q \leq p$, the smaller condition q is the stronger one. We stick to Goldstern's alphabetic convention and use lexicographically bigger symbols for stronger conditions.
- $E^\kappa_\lambda = \{\alpha \in \kappa : \mathrm{cf}(\alpha) = \lambda\}$.
- NS_κ is the nonstationary ideal on κ.
- The dual of an ideal I is the filter $\{A \subseteq \kappa : \kappa \setminus A \in I\}$ and vice versa.
- For an ideal I on κ and a positive set A (i.e., $A \notin I$), we set $I \restriction A$ to be the ideal generated by $I \cup \{\kappa \setminus A\}$.

We always assume that κ is a regular uncountable cardinal and that I is a $<\kappa$-complete filter on κ. Unless noted otherwise, we will also assume that I is normal.

We now recall the definitions of several games of length ω, played by the players empty and nonempty. We abbreviate "having a winning strategy for G" with "winning G" (as opposed to: "winning a specific run of G").

We now define four variants of the pressing down game (this game has been used, e.g., in [15]).

Definition 1.1.
- $\mathrm{PD}(I)$ is played as follows: Set $S_{-1} = \kappa$. At stage n, empty chooses a regressive function $f_n : \kappa \to \kappa$, and nonempty chooses S_n, an f_n-homogeneous subset of S_{n-1}. Empty wins the run of the game if $\bigcap_{n\in\omega} S_n \in I$.

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- $\mathrm{PD}^\emptyset(I)$ is played like $\mathrm{PD}(I)$, but empty wins the run if $\bigcap_{n\in\omega} S_n = \emptyset$.
- $\mathrm{PD}_\mathrm{e}(I)$ is played like $\mathrm{PD}(I)$, but empty can first choose S_{-1} to be an arbitrary I-positive set.
- $\mathrm{PD}^\emptyset_\mathrm{e}$ is defined analogously.

So we have four variants of the pressing down game, depending on two parameters: whether the winning condition for player nonempty is "$\neq\emptyset$" or "$\notin I$", and whether empty has the first move or not.

We now define vartiants of the Banach Mazur game:

Definition 1.2.
- $\mathrm{BM}(I)$ is played as follows: Set $S_{-1} = \kappa$. At stage n, empty chooses an I-positive subset X of S_{n-1}, and nonempty chooses an I-positive subset S_n of X. Empty wins the run if $\bigcap_{n\in\omega} S_n \in I$.
- The ideal game $\mathrm{Id}(I)$ is played just like $\mathrm{BM}(I)$, but empty wins the run if $\bigcap_{n\in\omega} S_n = \emptyset$.
- $\mathrm{BM}_\mathrm{ne}(I)$ is played just like $\mathrm{BM}(I)$, but nonempty has the first move.
- $\mathrm{Id}_\mathrm{ne}(I)$ is defined analogously.

More generally, we can define the Banach Mazur game $\mathrm{BM}(B)$ on a Boolean algebra B: The players choose decreasing (nonzero) elements $a_n \in B$, nonempty wins if there is some (nonzero) $b \in B$ smaller than all a_n. Then $\mathrm{BM}(I)$ is equivalent to the corresponding game $\mathrm{BM}(B_I)$ on the Boolean algebra $B_I = \mathfrak{P}(\kappa)/I$ (since I is σ-closed), the same holds for $\mathrm{BM}_\mathrm{ne}(I)$ and $\mathrm{BM}_\mathrm{ne}(B_I)$; we could equivalently use the completion $\mathrm{ro}(B_I)$ instead of B_I. Also the $\notin I$ versions of the pressing down game can be played modulo null sets, i.e., on the Boolean algebra B_I, in the obvious way. For the $\neq\emptyset$ versions of the games, the version played on B_I does not make sense.

In the $\notin I$ version, the pressing down and Banach Mazur games have natural generalizations to other lengths δ: At a limit stages γ, we use $\bigcap_{\alpha<\gamma} S_\alpha$ instead of $S_{\gamma-1}$, and empty wins a run iff this set is in I for any $\gamma < \delta$. (I.e., nonempty wins a run iff the run has length δ. So in this setting, the games defined above are the ones of length $\omega + 1$.) For the $\neq\emptyset$ versions of the games, lengths other than $\omega + 1$ seem less natural.

These games are also closely related to the cut and choose game introduced by Jech [11] (and its ancestor, the Ulam game):

Definition 1.3. The **cut and choose** game $\mathrm{c\&c}(B, \lambda)$ on a Boolean algebra B is played as follows: First empty chooses a nonzero element a_0 of B. At stage n, empty chooses a maximal antichain A_n below a_n of size at most λ, and nonempty chooses an element a_{n+1} from A_n. Nonempty wins the run if there is some nonzero b below all a_n.
$\mathrm{c\&c}(B, \infty)$ is played without restriction on the size of the antichains.

We can also define a set-partition version $\mathrm{c\&c}^\mathrm{set}(I, \lambda)$ of the game: Again, first empty chooses a positive set a_0; at stage n player empty partitions a_n into at most λ many I-positive pieces, and nonempty chooses a piece a_{n+1}. Empty wins the run if $\bigcap_{n\in\omega} a_n \in I$. However, this does not bring anything new:

- There can be at most κ many pieces, so $\mathrm{c\&c}^\mathrm{set}(I, \infty) = \mathrm{c\&c}^\mathrm{set}(I, \kappa)$.
- For $\lambda < \kappa$, $\mathrm{c\&c}^\mathrm{set}(I, \lambda)$ is equivalent to $\mathrm{c\&c}(B_I, \lambda)$.

- If I is nowhere κ-saturated,[1] then $\mathrm{c\&c}^{\mathrm{set}}(I,\kappa)$ is equivalent[2] to $\mathrm{BM}(I)$ (and therefore to $\mathrm{c\&c}(B_I,\infty)$, cf. 1.10).
- If I is κ^+-saturated, then $\mathrm{c\&c}^{\mathrm{set}}(I,\kappa)$, $\mathrm{c\&c}(B_I,\kappa)$, $\mathrm{PD_e}(I)$ and $\mathrm{BM}(I)$ are all equivalent, cf. 3.2 and 3.3.

We also consider the variant of $\mathrm{c\&c}^{\mathrm{set}}(I,\lambda)$ where empty wins if the intersection is empty (as opposed to null). However, then we have to allow empty to cut into arbitrary pieces, not just into I-positive ones, while empty still has to choose a positive piece. (Otherwise nonempty always wins, by fixing some α and picking the sets containing α.)

Definition 1.4.
- $\mathrm{c\&c}^{\mathrm{min}}(I,2)$ is played as follows: First, empty chooses some positive S_{-1}. At stage n, empty cuts S_{n-1} into two arbitrary pieces, and nonempty chooses an I-positive piece S_n.
- In $\mathrm{c\&c}^{\mathrm{min}}(I,{<}\kappa)$ empty cuts into less than κ many arbitrary pieces.
- $\mathrm{c\&c}^{\mathrm{min}}_{\mathrm{ne}}$ is defined as usual, i.e., $S_{-1}=\kappa$

We are interested in the existence of winning strategies:

Definition 1.5.
- We write $\mathfrak{b}(G)$ for "nonempty wins G" and $\mathfrak{a}(G)$ for "empty does not win G".
- The games G and H are equivalent, if $\mathfrak{b}(G)\leftrightarrow\mathfrak{b}(H)$ and $\mathfrak{a}(G)\leftrightarrow\mathfrak{a}(H)$.
- G is stronger than H, if $\mathfrak{b}(G)\rightarrow\mathfrak{b}(H)$ and $\mathfrak{a}(G)\rightarrow\mathfrak{a}(H)$.

We trivially get the following implications, see Figure 1:

Facts 1.6.
- $\mathfrak{b}(G)\rightarrow\mathfrak{a}(G)$ for all games.
- The Banach-Mazur game is stronger than the according pressing down game. E.g., $\mathrm{BM_{ne}}(I)$ is stronger than $\mathrm{PD}(I)$ etc.
- The $\notin I$ version is stronger than the $\neq\emptyset$ one. E.g., $\mathrm{BM}(I)$ is stronger than $\mathrm{Id}(I)$ etc.
- The version with empty choosing first is stronger. E.g., $\mathrm{BM}(I)$ is stronger than $\mathrm{BM_{ne}}(I)$ etc.
- $\mathrm{PD}^{\emptyset}_{\mathrm{e}}(I)$ is stronger than $\mathrm{c\&c}^{\mathrm{min}}(I,{<}\kappa)$, and $\mathrm{c\&c}^{\mathrm{min}}(I,{<}\kappa)$ is stronger than $\mathrm{c\&c}^{\mathrm{min}}(I,2)$.

In the rest of the section, we list a few well known (or otherwise obvious) facts. About BM and precipitous ideals[3] [10, 5, 8, 8]:

Facts 1.7.
- $\mathfrak{a}(\mathrm{Id}(I))$ is equivalent to "I is precipitous".
- $\mathfrak{a}(\mathrm{Id_{ne}}(I))$ is sometimes called "I is somewhere precipitous", and its failure "I is nowhere precipitous".
- A precipitous ideal implies that κ is measurable in an inner model.
- $\mathfrak{b}(\mathrm{BM}(\mathrm{NS}_{\aleph_2}\restriction E^{\aleph_2}_{\aleph_1}))$ is equiconsistent to a measurable.
- "$\mathrm{NS}_{\aleph_1}$ is precipitous" is also equiconsistent to a measurable.
- $\mathfrak{b}(\mathrm{Id}(I))$ implies $\kappa>2^{\aleph_0}$ and $E^{\kappa}_{\aleph_0}\in I$.
- $\mathfrak{a}(\mathrm{BM}(I))$ implies $E^{\kappa}_{\aleph_0}\in I$, and in particular $\kappa>\aleph_1$.

Some notes on PD (for normal ideals I,J):

[1] which is always the case if κ is a successor [9, 22.24]

[2] Instead of choosing a set X in the Banach Mazur game, empty can partition X into κ many pieces and add a single point to each piece so that the result is a partition of all of κ.

[3] these facts are of course true for I that are not neccessarily normal

$$
\begin{tikzcd}
 & \mathfrak{b}(\mathrm{BM}) \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathfrak{b}(\mathrm{Id}) \arrow[d] \arrow[dr] & \mathfrak{a}(\mathrm{BM}) \arrow[dl] \arrow[dr] & \mathfrak{b}(\mathrm{PD_e}) \arrow[dl] \arrow[d] \\
\mathfrak{a}(\mathrm{Id}) \arrow[dr] & \mathfrak{b}(\mathrm{PD}^{\emptyset}_{\mathrm{e}}) \arrow[d] & \mathfrak{a}(\mathrm{PD_e}) \arrow[dl] \\
 & \mathfrak{a}(\mathrm{PD}^{\emptyset}_{\mathrm{e}}) &
\end{tikzcd}
$$

FIGURE 1. The trivial implications (for empty moving first)

Facts 1.8.
- In the pressing down games, we can assume without loss of generality that nonempty chooses at stage n a set of the form $S_n = f_n^{-1}(\alpha_n) \cap S_{n-1}$ for some α_n.[4]
- PD is monotone in the following sense: if $J \supseteq I$, then $\mathrm{PD}(J)$ is stronger than $\mathrm{PD}(I)$. The same holds for $\mathrm{PD}^{\emptyset}$, but not for $\mathrm{PD_e}$ or $\mathrm{PD}^{\emptyset}_{\mathrm{e}}$ or for any of the Banach Mazur games.
- In particular, $\mathrm{PD}(I)$ is stronger than $\mathrm{PD}(\mathrm{NS}_\kappa)$ for all normal I.
- Just as in the case of BM, $\mathfrak{b}(\mathrm{PD}^{\emptyset})$ cannot hold for $\kappa = \aleph_1$ (cf. 5.2). But other than in the case of Id, the property $\mathfrak{a}(\mathrm{PD_e})$ has no consistency strength (cf. 3.1).

What is the effect of empty moving first?

Facts 1.9.
- For the Banach-Mazur games, the distinction whether empty has the first move or nonempty is a simple density effect: For example, nonempty wins $\mathrm{BM_{ne}}(I)$ iff there is some $S \in I^+$ such that nonempty wins $\mathrm{BM}(I \restriction S)$; similarly simple equivalences hold for empty winning; for characterizing BM in terms of $\mathrm{BM_{ne}}$; and for the $\not\models \emptyset$ version.
- We will see in Lemma 2.6 that this is not the case for the pressing down game.

The cut and choose game is related to the pressing down and Banach Mazur games [12, 4, 17, 18]. As usual, $\mathrm{ro}(B)$ denotes the completion of the Boolean algebra B, and $B_I = \mathfrak{P}(\kappa)/I$.

Facts 1.10.
- $\mathrm{c\&c}(B, \infty)$ is equivalent to $\mathrm{c\&c}(\mathrm{ro}(B), \infty)$; but $\mathrm{c\&c}(B, \lambda)$ will generally not be equivalent to $\mathrm{c\&c}(\mathrm{ro}(B), \lambda)$.
- $\mathrm{c\&c}(B, \infty)$ is equivalent to the Banach Mazur game on B.
- So in our case, $\mathrm{c\&c}(B_I, \infty)$ is equivalent to $\mathrm{BM}(I)$.
- $\mathrm{c\&c}(B_I, \kappa)$ is equivalent to $\mathrm{PD}(I)$, cf. 3.2. (However $\mathrm{c\&c}(\mathrm{ro}(B_I), \kappa)$ might be a stronger game.)

The set-partition version of the cut and choose game is also related to the pressing down and Banach Mazur games: If we assume that I is nowhere κ-saturated, then

[4]This of course means: PD is equivalent to the game where nonempty is restricted to moves of this form.

$$\begin{tikzcd}
\mathfrak{b}(\mathrm{BM}) \arrow[rr] \arrow[dr] \arrow[dd] & & \mathfrak{b}(\mathrm{PD_e}) \arrow[r] \arrow[dd] & \mathfrak{b}(\mathrm{PD}) \arrow[dd] \\
& \mathfrak{a}(\mathrm{BM}) \arrow[dd] & & \\
\mathfrak{b}(\mathrm{Id}) \arrow[rr] \arrow[dr] & & \mathfrak{b}(\mathrm{PD_e^{\emptyset}}) \arrow[r] & \mathfrak{b}(\mathrm{PD}) \arrow[d] \\
& \mathfrak{a}(\mathrm{Id}) \equiv \text{precip.} \arrow[rr] & & \infty\text{-semi precipitous}
\end{tikzcd}$$

FIGURE 2. Some properties stronger than ∞-semi precipitous

$\mathrm{BM}(I)$ is equivalent to $\mathrm{c\&c}(B_I, \kappa)$, i.e., to the game where nonempty partitions the current set into $\leq\kappa$ many positive pieces. In contrast, $\mathrm{PD_e}(I)$ is equivalent to the game where nonempty partitions the current set into $\leq\kappa$ many positive pieces with the additional requirement that the partition is maximal in the family of I-positive sets (i.e., for every positive set S there is an A in the partition such that $S \cap A$ is positive).

Winning strategies for games on a Boolean algebra B have close connections to the properties of B as Boolean algebra and as forcing notions, again see [12, 4, 17, 18]:

Facts 1.11.
- B having a σ-closed positive subset implies $\mathfrak{b}(\mathrm{BM}(B))$.
- $\mathfrak{b}(\mathrm{BM}(B))$ is also denoted by "B is strategically σ-closed" and implies that B is proper.
- $\mathfrak{a}(\mathrm{BM}(B))$ is equivalent to "B is σ-distributive".

It is not surprising that we will get stronger connections if we assume that the B has the form $B_I = \mathfrak{P}(\kappa)/I$ for a normal ideal I. We will mention only one example:

Fact 1.12. If B_I is proper, then $\mathfrak{a}(\mathrm{BM}(I))$ holds.

For a proof, see 3.5.

2. THE RESULTS

2.1. **Some Observations.** Some of the facts for precipitous ideals can be shown (with similar proofs) for PD, but there are of course strong differences as well:

Lemma 2.1.
(1) *$\mathfrak{b}(c\&c^{min}_{ne}(I,2))$ implies that κ is measurable in an inner model.*
(2) *So in particular, $\mathfrak{b}(PD^{\emptyset}(I))$ implies that as well.*
(3) *However, $\mathfrak{a}(PD_e(I))$ has no consistency strength. In particular, for $\kappa = \aleph_2$, $\mathfrak{a}(PD_e(I))$ is implied by CH for every I concentrated on $E^{\aleph_2}_{\aleph_1}$.*
(4) *$\mathfrak{b}(PD^{\emptyset}(I))$ implies $\kappa > 2^{\aleph_0}$ and that I is not concentrated on $E^{\kappa}_{\aleph_0}$.*

The proofs can be found in 5.5, 5.2 and 5.3.

In this paper, we are not interested in the property "empty does not win the pressing down game", since it has no consistency strength. Also, the effect of who moves first in Banach Mazur games is trivial. The remaining properties are pictured in Figure 2. All these properties are equiconsistent to a measurable (e.g., for $I = \mathrm{NS}_{\aleph_2} \restriction E^{\aleph_2}_{\aleph_1}$). In fact, they imply that I is ∞-semi precipitous, see Definition 4.1, which in turn implies that κ is measurable in an inner model. We claim that none of the implications can be reversed. In this paper, we will prove some stron instances of this claim by assuming larger cardinals: We show

- $\mathfrak{b}(\mathrm{PD_e})$ doe not imply precipitous, and
- $\mathfrak{a}(\mathrm{BM})$ does not imply $\mathfrak{b}(\mathrm{PD}^\emptyset)$.

We also claim that (consistently relative to a measurable)

- $\mathfrak{b}(\mathrm{Id})$ does not imply $\mathfrak{b}(\mathrm{PD})$,

but we do not give a proof here. With this claim (for which we assume cardinals larger than a measurable) it is then easy to check that no implication of Figure 2 kann be reversed.

In [14], Pauna and the authors showed that, assuming the consistency of a measurable, $\mathfrak{b}(\mathrm{PD}(I))$ does not imply $\mathfrak{b}(\mathrm{BM_{ne}}(I))$ for $I = \mathrm{NS}_{\aleph_2} \restriction E^{\aleph_2}_{\aleph_1}$. In fact, a slightly stronger statement holds (with a simpler proof):

Lemma 2.2. *It is equiconsistent with a measurable that $\mathfrak{b}(PD(I))$ holds (even for length ω_1) but $\mathfrak{a}(Id_{ne}(I)))$ fails for $I = \mathrm{NS}_{\aleph_2} \restriction E^{\aleph_2}_{\aleph_1}$.*

(For a proof, see 5.8.) Note that "$\mathfrak{a}(\mathrm{Id_{ne}}(I)))$ fails" just means that I is nowhere precipitous.

Of course, precipitous cannot generally imply a winning strategy for nonempty in any game, since precipitous is consistent with $\kappa \leq 2^\aleph_0$. However, we can get counterexampls for $\kappa > 2^{\aleph_0}$ as well: Just adding Cohens destroys any winning strategy for nonempty (for any ideal on $\aleph_2$), but preserves precipitous. So we get:

Lemma 2.3. *It is equiconsistent with a measurable that CH holds, $\mathrm{NS}_{\aleph_2} \restriction E^{\aleph_2}_{\aleph_1}$ is precipitous but $\mathfrak{b}(PD^\emptyset(J))$ fails for any normal ideals J on $\aleph_2$.*

2.2. **Large cardinals.** To see that not even $\mathfrak{a}(\mathrm{BM}(I))$ implies any winning strategy for nonempty, we assume CH and a $\aleph_3$-saturated ideal I on $\aleph_2$ concentrated on $E^{\aleph_2}_{\aleph_1}$. Saturation is preserved by small forcings, in particular by adding some Cohens, and saturation (together with CH) implies $\mathfrak{a}(\mathrm{BM}(I))$. So we get:

Lemma 2.4. *The following is consistent with CH plus an $\aleph_3$-saturated ideal on $\aleph_2$: CH holds, $\mathfrak{a}(Id(I))$ holds for some I on $\aleph_2$, but $\mathfrak{b}(PD^\emptyset(J))$ fails for any normal J on $\aleph_2$.*

See 5.11 and 5.12. (It seems very likely that saturation is not needed for this, but the construction might get considerably more complicated without it.)

As mentioned in Lemma 2.2 it is possible that $\mathfrak{b}(\mathrm{PD}(I))$ holds for a nowhere precipitous ideal, i.e., for an ideal such that $\mathfrak{a}(\mathrm{Id_{ne}}(I))$ fails. It seems harder to even get $\mathfrak{b}(\mathrm{PD_e}(I))$ for a nowhere precipitous ideal. Here we use a supercompact:

Theorem 2.5. *It is consistent with a supercompact that for $\kappa = \aleph_2$ there is a nowhere precipitous I such that $\mathfrak{b}(PD_e(I))$ holds (even for length ω_1).*

(See 6.1.)

Note that (as opposed to 2.3, 2.4) we just make a specific ideal non-precipitous, and we do not destroy all precipitous ideals. It seems very hard (and maybe impossible) to do better, and it is not clear whether we can avoid cardinals larger than a measurable. It is not known how to kill all precipitous ideals[5] on, e.g., $\aleph_1$

[5]Since we are only interested in normal ideals, it would be enough to kill all normal precipitous ideals. This doesn't help much, though; it is not known whether the existence of a precipitous ideal does imply the existence of a normal precipitous one. Recently Gitik [7, 6, 3] proved some interesting results in this direction.

with "reasonable" forcings.[6] And it might be even harder to do so while additionally preserving $\mathfrak{b}(\mathrm{PD_e}(I))$ for some ideals: By recent results by Gitik [6] (and later Ferber and Gitik [3]) a ∞-semi precipitous ideal does imply a normal precipitous ideal under in the absence of larger cardinals and under some cardinal arithmetic assumptions.

2.3. **Moving first.** Let us now investigate the effect of whether empty moves first.

If we compare G_{e} and H_{ne} for any games G and H, then these variants will be different for trivial reasons: For example, $\mathfrak{b}(\mathrm{BM_{ne}}(\mathrm{NS}_{\aleph_2}))$ does not imply $\mathfrak{b}(\mathrm{PD}^{\emptyset}_{\mathrm{e}}(\mathrm{NS}_{\aleph_2}))$: Let U be a normal ultrafilter on κ, Levy-collapse κ to $\aleph_2$, and let I_1 be the ideal generated by the dual of U (which is concentrated on $E^{\aleph_2}_{\aleph_1}$). Then nonempty wins $\mathrm{BM}(I_1)$ and therefore $\mathrm{BM_{ne}}(I_1 + E^{\aleph_2}_{\aleph_0})$ as well. But nonempty can never win $\mathrm{PD}^{\emptyset}_{\mathrm{e}}(I_1 + E^{\aleph_2}_{\aleph_0})$, since nonempty cannot win $\mathrm{PD}^{\emptyset}(E^{\aleph_2}_{\aleph_0})$.

So the games are very different (and for trivial reasons) when we change who has the first move. However, for the Banach Mazur game, who moves first results in a simple density effect, [7] as we have mentioned in 1.9. For example, $\mathfrak{b}(\mathrm{BM_{ne}}(I))$ holds iff $\mathfrak{b}(\mathrm{BM}(I \restriction S))$ holds for some positive S.

This is not the case for the pressing down games. Of course we still get:

- $\mathfrak{b}(\mathrm{PD_e}(I))$ holds iff $\mathfrak{b}(\mathrm{PD}(I \restriction S))$ holds for all $S \in I^+$.
- The same holds for $\mathrm{PD}^{\emptyset}_{\mathrm{e}}$.

But unlike the Banach Mazur case, we can have the following:

Lemma 2.6. *It is equiconsistent with a measurable that $\mathfrak{b}(PD(I))$ holds but $\mathfrak{b}(PD_e(I \restriction S))$ fails for all positive S, e.g., for $I = \mathrm{NS}_{\aleph_2}$.*

(See 5.8.) So in other words, $\mathfrak{b}(\mathrm{PD}(I))$ can hold but for all positive S there is a positive $S' \subseteq S$ such that $\mathfrak{b}(\mathrm{PD}(I \restriction S'))$ fails.

3. Empty not winning

Lemma 3.1.
- *CH implies $\mathfrak{a}(PD(I))$ for every I on $\aleph_2$ that is not concentrated on $E^{\aleph_2}_{\aleph_0}$.*
- *More generally, if $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$ and $\kappa^{\aleph_0} = \kappa$, then empty wins $PD(I)$ iff $E^{\kappa}_{>\omega} \in I$.*
- *So if I is concentrated on $E^{\kappa}_{>\omega}$ (and the same cardinal assumptions hold) then $\mathfrak{a}(PD_e(I))$ holds.*

Proof. Assume that I is concentrated on $E^{\kappa}_{\aleph_0}$. Just as in [5], it is easy to see that empty wins $\mathrm{PD}(I)$: For every $\alpha \in E^{\kappa}_{\aleph_0}$, let $(\mathrm{seq}(\alpha, n))_{n\in\omega}$ be a cofinal sequence in α. Let F_n map α to $\mathrm{seq}(\alpha, n)$. If empty plays F_n at stage n, then the intersection can contain at most one element.

So assume towards a contradiction that $E^{\kappa}_{>\omega} \notin I$ and that empty has a winning strategy for $\mathrm{PD}(I)$. The strategy assigns sets X_t and regressive function f_t to nodes t in the tree $T = \kappa^{<\omega}$ in the following way:

For $t = \langle\rangle$, set $X_{\langle\rangle} = \kappa$ and let $f_{\langle\rangle}$ be empty's first move. For $\alpha \in \kappa$, set $X_{(\alpha)} = f_t^{-1}(\alpha)$. Note that α is a valid response for nonempty iff $X_{(\alpha)}$ is positive.

[6]More specifically, it is not known whether large cardinals imply a precipitous ideal on $\aleph_1$, although Woodins are not enough, cf. [16].

[7]In games of length bigger than $\omega + 1$ however it does make a substantial difference who moves first at limits.

Generally, fix $t \in T$. We can assume by induction that one of the following cases hold:

- t corresponds to a partial run r_t with (positive) partial result X_t; then we set f_t to be empty's response to r_t.
- $X_t \in I$; then we set $f_t \equiv 0$.

In both cases we set $X_{t^\frown\alpha} = X_t \cap f_t^{-1}(\alpha)$.

Let b be a branch of T (i.e., $b \in \kappa^\omega$). We set $X^b = \bigcap_{n\in\omega} X^{b\restriction n}$.

Assume that b corresponds to a run of the game; this is the case iff $X_{b\restriction n}$ is I-positive for all n. Then $X^b \in I$, since empty uses the winning strategy. If b does not correspond to a run, then $X^b \in I$ as well. So

$$X^b \in I \text{ for all branches } b \tag{1}$$

X^b and X^c are disjoint for different branches b, c; and for all $\gamma \in \kappa$ there is exactly one branch b_γ such that $\gamma \in X^{b_\gamma}$. We assume $\gamma \neq 0$ from now on. By definition, for all n

$$f_{b_\gamma\restriction n}(\gamma) = b_\gamma(n)$$

Since $f_{b_\gamma\restriction n}$ is regressive, $b_\gamma(n) < \gamma$ for all $n \in \omega$. In other words, $b_\gamma \in \gamma^\omega$.

Fix an injective function $\phi : \kappa^\omega \to \kappa$. Since $\gamma^{\aleph_0} < \kappa$ for $\gamma < \kappa$, we can find a club C such that

$$\phi''\gamma^\omega \subseteq \gamma \text{ for all } \gamma \in C \cap E^\kappa_{>\omega}.$$

This defines a regressive function $g : C \cap E^\kappa_{>\omega} \to \kappa$ by $g(\gamma) = \phi(b_\gamma)$. Since I is normal and does not contain $E^\kappa_{>\omega}$, there is a positive set S and a $\zeta \in \kappa$ (or equivalently a branch b of T) such that $g(\gamma) = \zeta$, i.e., $b_\gamma = b$ for all $\gamma \in S$. This implies that $S \subseteq X^b$ is positive, a contradiction to (1). $\square$

Lemma 3.2. *If I is normal, then PD_e is equivalent to $c\&c(B_I, \kappa)$.*

Proof. A regressive function defines a maximal antichain in B_I of size at most κ. On the other hand, let A be a maximal antichain of size $\lambda \leq \kappa$. We can choose pairwise disjoint representatives $(S_i)_{i\in\lambda}$ for the elements of A, and define

$$f(\alpha) = \begin{cases} 1+i & \text{if } \alpha \in S_i \text{ and } 1+i < \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

$f^{-1}(0) \in I$. (Otherwise there is an S_i in A such that $T = S_i \cap f^{-1}(0) \in I^+$, pick $\alpha \in T \setminus (1+i+1)$, contradiction.) So the partition A is equivalent to the regressive function f. $\square$

Together with 1.10 we get:

Corollary 3.3. *If I is normal and κ^+-saturated, then $BM(I)$ and $PD_e(I)$ are equivalent. The same holds for $BM_{ne}(I)$ and $PD(I)$.*

If I is κ^+-saturated, then it is precipitous, i.e., $\mathfrak{a}(\mathrm{Id}(I))$ holds [9, 22.22]. However, I be concentrated on $E^\kappa_{\aleph_0}$ (for example, κ could be $\aleph_1$), which negates $\mathfrak{a}(\mathrm{BM}(I))$. However, with Lemma 3.1 we get:

Corollary 3.4. *If I is κ^+-saturated, $\kappa^{\aleph_0} = \kappa$, $\lambda^{\aleph_0} < \kappa$ for all $\lambda < \kappa$, and I is normal and concentrated on $E^\kappa_{>\aleph_0}$, then $\mathfrak{a}(BM(I))$ holds.*

In the rest of the section, we show that properness implies $\mathfrak{a}(\mathrm{BM}(I))$. This is not needed for the rest of the paper.

For any Boolean algebra B, $\mathfrak{b}(\mathrm{BM}(B))$ implies that B is proper (as a forcing notion), cf. e.g. [11, Thm. 7]. For Boolean algebras of the form $B_I = \mathfrak{P}(\kappa)/I$ we also get:

Lemma 3.5. *Assume $\kappa > 2^{\aleph_0}$. If B_I is proper then $\mathfrak{a}(BM(I))$ holds.*

Normality of I is not needed, just $<\kappa$-completeness.

Proof. Assume towards a contradiction that τ is a winning strategy for empty. Let $p_0 \in I^+$ be empty's first move according to τ. Pick $N \prec H(\chi)$ countable containing I and τ (and therefore p_0), and let $q \leq p_0$ be N-generic. In other words, if $\mathcal{D} \in N$ is a predense subset of I^+, then q is (mod I) a subset of $\bigcup(\mathcal{D} \cap N)$. Therefore

$$\mathcal{X} = q \cap \bigcap\{\bigcup(\mathcal{D} \cap N) : \mathcal{D} \subseteq I^+ \text{ is predense and } \mathcal{D} \in N\},$$

is positive. We set

$$\mathcal{Y} = \bigcup\{\bigcap_{n\in\omega} A_n : (\forall n \in \omega)\, A_n \in N \cap I^+,\ \bigcap_{n\in\omega} A_n \in I\}$$

$Y \in I$, since $|[N]^{\aleph_0}| < \kappa$. So we can pick some

$$\delta^* \in \mathcal{X} \setminus Y.$$

We now construct a run of the game such that every initial segment is in N. Assume that we already know the inital segment of the first $n-1$ stages, and that this segment is in N. Then empty's move A_n given by τ is in N as well. We further assume that $\delta^* \in A_n$. (This is true for $n = 0$, since $\delta^* \in q \leq p_0$.) For any I-positive $B \subseteq A_n$ let empty's response be $f(B)$. The set

$$D = \{\kappa \setminus A_n\} \cup \{f(B) : B \subseteq A_n \text{ positive}\}$$

is dense in I^+ and is in N. Since $\delta^* \in \mathcal{X}$, $\delta^* \in \bigcup(D \cap N)$, i.e. there is some $B \in N$ such that $\delta^* \in f(B)$. Let B be nonempty's move.

So δ^* will be in the intersection $Z = \bigcap A_n$, and since empty wins the run, $Z \in I$. Since each A_n is in N, we get $Z \subseteq \mathcal{Y}$. This contradicts $\delta^* \in Z$. $\square$

4. ∞-SEMI PRECIPITOUS IDEALS

Definition 4.1. A κ-complete ideal I on κ is called (normally) ∞-semi precipitous, if there is some partial order P which forces that there is a (normal) wellfounded, nonprincipal, κ-complete V-ultrafilter containing the dual of I.

Donder, Levinski [2] introduced the notion of λ-semi precipitous, and Ferber and Gitik [3] extended the notation to ∞-semi precipitous. Another name, "weakly precipitous", is used for this notion in [1]. However, Jech uses the term "weakly precipitous" for another concept, cf. [12, 2].

We will see in Lemma 5.5 that $\mathfrak{b}(\mathrm{PD}^{\emptyset}(I))$ implies that I is normally ∞-semi precipitous. This will establish the consistency strength of $\mathfrak{b}(\mathrm{PD}^{\emptyset}(I))$:

Lemma 4.2. *If there is an ∞-semi precipitous ideal on κ, then κ is measurable in an inner model.*

This is of course no surprise: the proof is a simple generalization of the proof [8, Theorem 2] for precipitous; Jech and others have used in fact very similar generalizations. (E.g., in [12] it is shown more or less that pseudo-precipitous ideals are ∞-semi precipitous.)

Proof. We assume that there is a forcing P and a name $\underset{\sim}{D}$ for the V-generic filter. In particular:

(2) P forces that in $V[G]$ there is an elementary embedding $j : V \to M$ for some transitive class M in $V[G]$.

If we are only interested in consistency strength, we can use Dodd-Jensen core model theory as a black-box: (2) is equiconsistent to a measurable cardinal, which follows immediately from [9, 35.6] and the remark after 35.14: $K^V = K^{V[G]}$, and there is a measurable iff there is an elementary embedding $j : K \to M$ (which also implies $M = K$). However, this only tells us that there is some ordinal which is measurable in an inner model, and not that this ordinal is indeed κ.

To see this, we can either use more elaborate core model theory (as pointed out by Gitik, cf. [19, 7.4.8,7.4.11]). Alternatively, we can just slightly modify the proof of [8, Theorem 2] (which can also be found in [9, 22.33]). We will do that in the following: Let K be the class of strong limit cardinals μ such that $\mathrm{cf}(\mu) > \kappa$. Let $(\gamma_n)_{n\in\omega}$ be an increasing sequence in K such that $|K \cap \gamma_n| = \gamma_n$. Set $A = \{\gamma_n > n \in \omega\}$ and $\lambda = \sup(A)$.

By a result of Kunen, it is enough to show the following:

(3) There is (in V) an iterable, normal, fine $L[A]$-ultrafilter W such that every iterated ultrapower is wellfounded.

We have a name $\underset{\sim}{D}$ for the V-generic filter. $\underset{\sim}{D}$ does not have to be normal, but there is some $p_0 \in P$ and $\alpha_0 \geq \kappa$ such that p_0 forces that $[\mathrm{Id}] = \alpha_0$. We set

$$\mathcal{J} = \{x \subseteq \kappa : \, p_0 \Vdash x \notin \underset{\sim}{D}\}, \text{ and}$$
$$U = \{x \in \mathfrak{P}(\kappa) \cap L[A] : \, x \notin \mathcal{J}\}.$$

U is generally not normal, but the normalized version of U will be as required.

U is an $L[A]$ ultrafilter: Let $x \subseteq \kappa$ be in $L[A]$. We have to show: x or $\kappa \setminus x$ are in $\mathcal{J}$.

- There is a formula φ and a finite $E \subseteq \kappa \cup K$ such that (in $L[A]$) $\alpha \in x$ iff $\alpha < \kappa$ and $\varphi(\alpha, E, A)$.
- Assume G is P-generic over V and contains p_0. $[\mathrm{Id}] = \alpha_0$, so $x \in \underset{\sim}{D}[G]$ iff $\alpha_0 \in j(x)$.
- By elementarity (in $V[G]$) $\alpha_0 \in j(x)$ iff $j(L[A])$ thinks that $\varphi(\alpha_0, j(E), j(A))$. But $j(\mu) = \mu$ for every $\mu \in K$.
- So we get $x \in \underset{\sim}{D}[G]$ iff (in $L[A]$) $\varphi(\alpha_0, E, A)$ holds, independently of G (provided G contains p_0). In other words, if there is some generic G such that $x \in \underset{\sim}{D}[G]$, then $x \in \underset{\sim}{D}[G]$ for all generic G (containing p_0); i.e. p_0 forces that $x \in \underset{\sim}{D}[G]$; i.e. $\kappa \setminus x \in \mathcal{J}$.
- Assume that x is not in $\mathcal{J}$. Then there is some $q \leq p$ forcing that $x \in \underset{\sim}{D}$. So $\kappa \setminus x \in \mathcal{J}$.

U is $<\kappa$-complete, fine and wellfounded: Pick $\lambda < \kappa$ and $(x_\alpha)_{\alpha\in\lambda}$ in $L[A]$ such that each $x_\alpha \in U$. Then p_0 forces that $\kappa \setminus x_\alpha \notin \underset{\sim}{D}$, and therefore that $\bigcup \kappa \setminus x_\alpha \notin \underset{\sim}{D}$ (since $\underset{\sim}{D}$ is a $<\kappa$-complete ultrafilter).

This also shows that (in V) the intersection of $\aleph_0$ many U-elements is nonempty; which implies that every iterated ultrapower is wellfounded (provided iterability).

U is iterable: Let (in $L[A]$) $(x_\alpha)_{\alpha\in\kappa}$ be a sequence of subsets of κ. Let G be P-generic over V and contain p_0. In $V[G]$, $x_\alpha \in \underaccent{\tilde}{D}[G]$ iff $\alpha_0 \in j(x_\alpha)$. The sequence $(j(x_\alpha))_{\alpha\in\kappa}$ is in $L[j(A)]$, and therefore also the set $\{\alpha \in \kappa : \alpha_0 \in j_G(x_\alpha)\}$. But $L[j(A)] = L[A]$.

normalizing: Since we now know that U is wellfounded, we know that there is some $f : \kappa \to \kappa$ in $L[A]$ representing κ. Set $W = f_*(U)$. Then W is as required. $\square$

The following follows easily from Kunen's method of iterated ultrapowers:

Lemma 4.3. *Assume $V = L[U]$, where U is a normal ultrafilter on κ. Let V' be a forcing extension of V and $D \in V'$ a normal, wellfounded V-ultrafilter on κ. Then $D = U$.*

(See, e.g., [14, 4.3] for a proof.) This implies:

Corollary 4.4. *In $L[U]$, the dual of U is the only normal precipitous ideal on κ; and every ideal on κ that is normally ∞-semi precipitous is a subideal of the dual of U.*

We will also need the following:

Lemma 4.5. *If I is a $<\kappa$-complete ideal, P a κ-cc forcing notion, and* cl(I) *the P-name for the closure of I in $V[G]$, then P preserves the following properties: I is precipitous, I is not precipitous, and I is nowhere precipitous.*

Proof. This has been known for a long time, cf. e.g. [13]: "not precipitous" is equivalent to the existence of a decreasing sequence of functionals starting at some positive set S_0 (this corresponds to: S forces that there is an infinite decreasing sequence in the ultrapower, the sequence of functionals witnesses this). A κ-cc forcing preserves maximality (below S_0) of an antichain in B_I, and therefore the decreasing sequence of functionals. "Nowhere precipitous" is equivalent to the existence of a decreasing sequence of functionals starting with κ, which again is preserved by P. $\square$

5. Nonempty winning

Let us assume that nonempty has a winning strategy in $\mathrm{PD}^{\emptyset}(I)$ (or a similar game such as $\mathrm{PD}(I)$). A valid sequence is a finite initial sequence of a run of the game $\mathrm{PD}^{\emptyset}$, where nonempty uses his strategy. So a valid sequence w has the form $(f_0, \alpha_0, f_1, \alpha_1, \dots, f_{n-1}, \alpha_{n-1})$, where f_i is a regressive function and α_i the value chosen by the strategy. In particular $S_i = \bigcap_{j\leq i} f_j^{-1}(\alpha_j)$ is I-positive for each $i < n$. We set

$$A(w) = S_{n-1} = \bigcap_{j<n} f_j^{-1}(\alpha_j).$$

Definition 5.1. P^* is the set of valid sequences ordered by extension. (A longer sequence is stronger, i.e., smaller in the P^*-order.)

So if $w < v$, then $A(w) \subseteq A(v)$. If $w_0 > w_1 > w_2 > \dots$ is an infinite decreasing sequence in P^*, then $\bigcup_{i\in\omega} w_i$ represents a run of the game, so the result $\bigcap_{i\in\omega} A(w_i)$ has to be nonempty (or even positive in the case of a PD-strategy).

Lemma 5.2. *$\mathfrak{b}(PD^{\emptyset})$ implies $\kappa > 2^{\aleph_0}$.*

Actually, we can even restrict nonempty to play functions $f : \kappa \to \{0,1\}$. In other words, it is enough to assume $\mathfrak{b}(\mathrm{c\&c}^{\min}(I,2))$, cf. Definition 1.4.

Proof. The proof is the same as [5, §1]: We assume otherwise and identify κ with a subset X of $[0,1]$ without a perfect subset. We claim:

(4) For all $w \in P^*$ and $n \in \omega$ there are disjoint open intervals I_1 and I_2 of length $\leq 1/n$ and $w_1, w_2 < w$ such that $A(w_1) \subseteq I_1$ and $A(w_2) \subseteq I_2$.

Assume that (4) fails for some v_0 and n_0. Given $v < v_0$ and $n > n_0$, we fix a partition of $[0,1]$ into n many open intervals of length $1/n$ and the (finite) set of endpoints. By splitting $A(v)$ $n+1$ many times, empty can guarantee that $A(w)$ has to be subset of one of the intervals for some $w < v$. Since (4) fails, there has to be for each n a fixed element $I(n)$ of the partition such that for all $v < v_0$ there is a $w < v$ with $A(w) \subseteq I(n)$. $\bigcap I(n)$ can contain at most one point x, so the empty player can continue v_0 by first splitting into $\{x\}$ and $A(v_0) \setminus \{x\}$; and then extending each v_{n-1} to v_n such that $A(v_n) \subseteq I(n)$. Then the intersection is empty. This shows (4).

So we can fix an order preserving function ψ from $2^{<\omega}$ to P^* such that $A(\psi(s^\frown 0))$ and $A(\psi(s^\frown 1))$ are separated by intervals of length $\leq 1/|s|$ for all $s \in 2^{<\omega}$. Then every $\eta \in 2^\omega$ is mapped to a run of the game, and since nonempty wins, there is some $r_\eta \in \bigcap_{n\in\omega} A(\psi(\eta \restriction n))$. This defines a continuous, injective mapping from 2^ω into X and there fore a perfect subset of X. $\square$

Clearly $\mathfrak{a}(\mathrm{PD})$ fails if I is concentrated on $E^\kappa_{\aleph_0}$, and this was used in [5] to show that in this case $\mathfrak{b}(\mathrm{Id}_{\mathrm{ne}}(I))$ fails as well. A similarly easy proof gives:

Lemma 5.3. *$\mathfrak{b}(PD^\emptyset(I))$ fails if I is concentrated on $E^\kappa_{\aleph_0}$.*

Proof. Assume otherwise. Fix for each $\alpha \in E^\kappa_{\aleph_0}$ a normal cofinal sequence $(\mathrm{seq}(\alpha,n))_{n\in\omega}$, and let $g_i : \kappa \to \kappa$ map α to $\mathrm{seq}(\alpha, i)$. We first show a variant of (4):

(5) For all w there are $v_1, v_2 \leq w$ in P^* such that $A(v_1) \cap A(v_2) = \emptyset$.

Assume otherwise. Then for each i there is a fixed β_i such that nonempty responds with β_i whenever empty plays g_i in any $v \leq w$. Set $\delta = \sup\{\beta_i : i \in \omega\}$, and let empty play the following response to w:

$$f(\alpha) = \begin{cases} 0 & \text{if } \alpha \leq \delta, \\ \min\{n : \mathrm{seq}(\alpha, n) > \delta\} & \text{otherwise.} \end{cases}$$

If nonempty responds to f with m, then empty can play g_m as next move, nonempty has to respond with $\beta_m < \delta$, but

$$g_m^{-1}(\beta_m) = \{\alpha : \mathrm{seq}(\alpha, m) = \beta_m\}$$

is disjoint to $f^{-1}(m)$, a contradiction. This shows (5).

Now fix $N \prec H(\chi)$ of size less than κ containing the strategy as well as all g_n and such that $N \cap \kappa = \delta \in E^\kappa_{\aleph_0}$. We define a sequence $w_0 > w_1 > \cdots$ in P^* such that each w_i is in N: Using (5) in N, we get a $w_0 \in N \cap P^*$ such that $\delta \notin A(w_0)$. Given w_{n-1}, let $w_n \in N$ be the continuation where empty played the regressive function

$$f_n(\alpha) = \begin{cases} 0 & \text{if } \alpha < \mathrm{seq}(\delta, n) \\ g_n(\alpha) & \text{otherwise.} \end{cases}$$

(Note that $\mathrm{seq}(\delta, n) < \delta$ is in N for all n.) Assume that $\nu \in \bigcap_{n\in\omega} A(w_n)$. Then $\nu \geq \mathrm{seq}(\delta, n)$ for all n, so $\nu \geq \delta$. On the other hand, $g_n(\nu) \in N$ for all n, so $\nu \leq \delta$. But $\delta \notin A(w_0)$, a contradiction. $\square$

Of course this shows the following: $\mathfrak{b}(\mathrm{PD}^\emptyset(I))$ implies $\mathfrak{b}(\mathrm{PD}^\emptyset(I \restriction E^\kappa_{>\aleph_0}))$ (since empty can just cut κ into $E^\kappa_{\aleph_0}$ and $E^\kappa_{>\aleph_0}$ as a first move).

Recall that $\mathfrak{b}(\mathrm{PD}^\emptyset(I))$ for any I implies $\mathfrak{b}(\mathrm{PD}^\emptyset(\mathrm{NS}))$ (due to monotonicity). So the last lemma gives:

Corollary 5.4. *$\mathfrak{b}(PD^\emptyset(I))$ is equivalent to $\mathfrak{b}(PD^\emptyset(I \restriction E^\kappa_{>\aleph_0}))$ and implies $\mathfrak{b}(PD^\emptyset(\mathrm{NS}))$ and $\mathfrak{b}(PD^\emptyset(\mathrm{NS} \restriction E^\kappa_{>\aleph_0}))$.*

Lemma 5.5. *$\mathfrak{b}(PD^\emptyset(I))$ implies that I is normally ∞-semi precipitous.*
$\mathfrak{b}(c\&c^{min}(I, {<}\kappa))$ implies that I is ∞-semi precipitous.

Proof. We define the P^*-name $\underaccent{\sim}{U}$ by $X \in \underaccent{\sim}{U}$ iff $X \supseteq A(w)$ for some $w \in G_{P^*}$.

- P^* forces that $\underaccent{\sim}{U}$ is a V-ultrafilter: Given any $w \in P^*$ and $X \in V$, player empty can respond to w by cutting into X and $A(w) \setminus X$.
- In the c&c$^{\min}$ case, P^* forces that $\underaccent{\sim}{U}$ is ${<}\kappa$-complete: Assume that (in V) X is the disjoint union of $(X_i)_{i\in\lambda}$, $\lambda < \kappa$. Then empty can responds to w by cutting into $\{X_i : i \in \lambda\} \cup \{A(w) \setminus X\}$.
- In the case of PD, P^* forces that $\underaccent{\sim}{U}$ is V-normal: If $f \in V$ is regressive, then empty can play f as response to any w.
- P^* forces that $\underaccent{\sim}{U}$ is wellfounded: Assume towards a contradiction that w forces that $(\underaccent{\sim}{f}_n)_{n\in\omega}$ are functions (in V) from κ to the ordinals such that
$$\underaccent{\sim}{A}_n = \{\alpha : \underaccent{\sim}{f}_{n+1}(\alpha) < \underaccent{\sim}{f}_n(\alpha)\}$$
is in $\underaccent{\sim}{U}$ for all $n \in \omega$. Set $w_{-1} = w$. Assume that we already have w_n (for $n \geq -1$). Pick some $w'_{n+1} < w_n$ deciding $\underaccent{\sim}{f}_{n+1}$ to be some $f'_{n+1} \in V$. So w'_{n+1} forces that $X_{n+1} := \bigcap_{l\leq n+1} \underaccent{\sim}{A}_l = \bigcap_{l\leq n+1} A'_l$ (a set in V) is in $\underaccent{\sim}{U}$. In particular, there is some w_{n+1} stronger than w'_{n+1} such that $A(w_{n+1}) \subseteq X_{n+1}$. The sequence $(w_n)_{n\in\omega}$ corresponds to a run of the game. Since nonempty follows the strategy, there is some $\alpha \in \bigcap_{n\in\omega} A(w_n)$. w_{n+1} forces $\alpha \in X_{n+1}$, i.e., $f'_{n+1}(\alpha) < f'_n(\alpha)$. This gives an infinite decreasing sequence, a contradiction. $\square$

Together with 4.4, we get:

Corollary 5.6. *In $L[U]$, nonempty does not win $PD^\emptyset(\mathrm{NS}_\kappa \restriction S)$ for any $S \notin U$. In particular, $\mathfrak{b}(PD(\mathrm{NS}_\kappa))$ holds (even for the game of length κ), but $\mathfrak{b}(PD^\emptyset_e(\mathrm{NS}_\kappa \restriction S))$ fails for every stationary S. Also, and $\mathfrak{a}(Id(\mathrm{NS}_\kappa \restriction S))$ fails, i.e., NS_κ is nowhere precipitous.*

We can use a Levy Collapse to reflect this situation down to, e.g., $\aleph_2$. We first list some properties of the Levy collapse. Assume that κ is inaccessible, $\theta < \kappa$ regular, and let $Q = \mathrm{Levy}(\theta, <\kappa)$ be the Levy collapse of κ to θ^+: A condition $q \in Q$ is a function defined on a subset of $\kappa \times \theta$, such that $|\mathrm{dom}(q)| < \theta$ and $q(\alpha, \xi) < \alpha$ for $\alpha > 1, (\alpha, \xi) \in \mathrm{dom}(q)$ and $q(\alpha, \xi) = 0$ for $\alpha \in \{0, 1\}$. Given $\alpha < \kappa$, define $Q_\alpha = \{q : \mathrm{dom}(q) \subseteq \alpha \times \theta\}$ and $\pi_\alpha : Q \to Q_\alpha$ by $q \mapsto q \restriction (\alpha \times \theta)$. The following is well known:

- If $q \Vdash p \in G$, then $q \leq p$ (i.e. $\leq^*$ is the same as $\leq$).

- Q is κ-cc and $<\theta$-closed.
- In particular, if p forces that $C \subseteq \kappa$ is club, then there is a club $C_0 \in V$ such that p forces $C_0 \subseteq C$. The ideal generated by NS^V_κ in $V[G]$ is $NS^{V[G]}_\kappa$.

We also need the following simple fact (see, e.g., [14, 6.2] for a proof):

$$\text{(6)}\qquad \begin{array}{l}\text{Let } I \text{ be a normal ideal concentrated on } E^\kappa_{\geq\theta}, \text{ let } T \text{ be } I\text{-positive,}\\ p \in Q \text{ and } p_\alpha \leq p \text{ for all } \alpha \in T. \text{ Then there is an } I\text{-positive } T' \subseteq T\\ \text{and a } q \leq p \text{ such that } \pi_\alpha(p_\alpha) = q \text{ for all } \alpha \in T'.\end{array}$$

So in particular, every $q' \leq q$ is compatible with p_α for all but boundedly many $\alpha \in T'$.

We will also use:

Lemma 5.7. *Let κ be inaccessible and $T \subset \kappa$ be stationary. The Levay collapse preserves $\neg\mathfrak{b}(PD^\emptyset(\mathrm{NS}_\kappa \restriction T))$. The same holds for PD.*

Proof. Assume towards a contradiction that q forces that nonempty does have a winning strategy in $V[G]$. We describe a winning strategy in V: Assume empty plays f_0 (in $[V]$). Let $q_0 \leq q$ decide that in $V[G]$ nonempty chooses α_0 as response to f_0 according to the winning strategy in $V[G]$. So q_0 forces that $f_0^{-1}(\alpha_0) \cap T$ is stationary, therefore $f_0^{-1}(\alpha_0) \cap T$ is stationary in V. Generally, let $q_n \leq q_{n-1}$ decide that nonempty plays α_n as response to f_n. Since Q is σ-closed, there is a $q_\omega < q_n$ for all n. So q_ω forces that $\bigcap f_n^{-1}(\alpha_n) \cap T$ is stationary. $\square$

Starting with $L[U]$ and using a Levy collapse we get:

Corollary 5.8. *Consistently relative to a measurable, $\mathfrak{b}(PD(\mathrm{NS}_{\aleph_2}))$ holds (even for length $\aleph_1$) but $\mathfrak{b}(PD^\emptyset_e(\mathrm{NS}_{\aleph_2} \restriction S))$ fails for every stationary S, and $\mathrm{NS}_{\aleph_2}$ is nowhere precipitous.*

Proof. Assume $V = L[U]$ and let $Q = \mathrm{Levy}(\aleph_1, {<}\kappa)$ be the Levy collapse of κ to $\aleph_2$.

To see that $\mathrm{NS}_{\aleph_2}$ is forced to be nowhere precipitous, note that $<\kappa$-cc implies $\mathrm{cl}^{V[G]}(\mathrm{NS}^V_\kappa) = \mathrm{NS}^{V[G]}_\kappa$ and use 4.5.

In $V[G]$, $\mathrm{cl}^{V[G]}(U)$ is a normal filter such that the family of positive sets has a σ-closed dense subset [5]. Let I be the dual ideal. So nonempty wins $\mathrm{BM}(I)$, and therefore $\mathrm{PD_e}(I)$ and $\mathrm{PD}(\mathrm{NS}_\kappa)$ (even of length $\aleph_1$).

It remains to be shown that $\mathfrak{b}(\mathrm{PD}^\emptyset_\mathrm{e}(\mathrm{NS}_{\aleph_2} \restriction S))$ fails in $V[G]$ for all stationary S. Assume towards a contradiction that some p forces that $\underset{\sim}{S}$ is stationary and $\mathfrak{b}(\mathrm{PD}^\emptyset(\mathrm{NS}_\kappa \restriction S'))$ holds for all stationary $S' \subseteq \underset{\sim}{S}$. According to 5.4 we can assume $\underset{\sim}{S} \subseteq E^{\aleph_2}_{\aleph_1}$. Set

$$T_0 = \{\alpha \in \kappa : p \not\Vdash \alpha \notin \underset{\sim}{S}\}$$

$T_0 \subseteq E^\kappa_{\geq\aleph_1}$ is stationary. Fix some stationary $T \subseteq T_0$ not in U; and for $\alpha \in T$ fix some $p_\alpha \leq p$ forcing $\alpha \in \underset{\sim}{S}$. Apply (6) to T, the nonstationary ideal and $(p_\alpha)_{\alpha\in T}$. This results in $q \leq p$ and $T' \subseteq T$ stationary.

$$\text{(7)}\qquad q \Vdash S' := T' \cap \underset{\sim}{S} \text{ is stationary.}$$

Otherwise some $q_1 \leq q$ forces that S' is nonstationary. Then there is in V a club C and a $q_2 \leq q_1$ forcing that $S' \cap C = \emptyset$. Pick $\alpha \in T' \cap C$ such that p_α and q_2 are compatible. Then $q_3 \leq p_\alpha, q_2$ forces that $\alpha \in T' \cap C \cap \underset{\sim}{S}$, a contradiction. This shows (7).

By our assumption, p forces that nonempty wins $\mathrm{PD}^{\emptyset}(\mathrm{NS} \restriction S')$. But $\mathfrak{b}(\mathrm{PD}^{\emptyset}(\mathrm{NS}_{\kappa} \restriction T'))$ fails in V (since $T' \subset T$ and $T \notin U$), therefore $\mathfrak{b}(\mathrm{PD}^{\emptyset}(\mathrm{NS}_{\aleph_2} \restriction T'))$ fails in $V[G]$ according to 5.7, and by monotonicity $\mathfrak{b}(\mathrm{PD}^{\emptyset}(\mathrm{NS}_{\aleph_2} \restriction S'))$ fails as well, a contradiction. □

We will now force nonempty not to win PD. For simplicity we will assume CH and look at $\kappa = \aleph_2$. It turns out that it is enough to add $\aleph_1$ many Cohen reals (actually, many similar forcings also work). First we need another variant of (4) or (5):

Lemma 5.9. *Assume CH and $\mathfrak{b}(PD^{\emptyset}(\mathrm{NS}_{\aleph_2}))$. For each $v \in P^*$ there are $F'(v) \leq v$ and $F''(v) \leq v$ such that $A(F'(v))$ and $A(F''(v))$ are disjoint.*

(We can choose $F'(v)$ and $F''(v)$ to be immediate successors of v, i.e. we just have to choose two regressive functions f' and f'' as empty's moves.)

Proof. We fix an injection $\phi : [\aleph_2]^{\aleph_0} \to \aleph_2$. Let $S = C \cap E^{\aleph_2}_{\omega_1}$ (for some clubset C) consist of ordinals α such that $\phi''[\alpha]^{\aleph_0} \subseteq \alpha$. For each $\alpha \in S$, pick a normal cofinal sequence $\gamma^{\alpha} : \omega_1 \to \alpha$. Set $g_i(\alpha) = \phi(\{\gamma^{\alpha}(j) : j \leq i\})$; and $g_i(\alpha) = 0$ for $\alpha \notin S$. So for all $i \in \omega_1$, g_i is a regressive function. If $\alpha \neq \beta$ then $g_i(\alpha) \neq g_i(\beta)$ for some i; and $g_i(\alpha) \neq g_i(\beta)$ implies $g_j(\alpha) \neq g_j(\beta)$ for all $j > i$.

Let $x(i)$ be the strategy's response to $v^\frown g_i$, and pick some $\gamma(i)$ such that $g_i(\gamma(i)) = x(i)$.

Case A: There are some $i < j$ such that $\phi^{-1}x(i)$ is not inital segment of $\phi^{-1}x(j)$. In this case, set $f' = g_i$ and $f'' = g_j$: If $\alpha \in g_i^{-1}(x(i))$ and $\beta \in g_j^{-1}(x(j))$, then $\gamma^{\alpha} \restriction i \neq \gamma^{\beta} \restriction i$, so in particular $\alpha \neq \beta$.

Case B: Otherwise, let y be the union of $\phi^{-1}x(i)$, and set $\tilde{\gamma} = \sup(y)$. For all $\alpha \neq \tilde{\gamma}$, there is some $k(\alpha) \in \omega_1$ such that $g_{k(\alpha)}(\alpha) \neq g_{k(\alpha)}(\tilde{\gamma})$. Let $\tilde{k}$ be the strategy's response to k. Then set $f' = k$ and $f'' = g_{\tilde{k}}$. Assume $g_{\tilde{k}}(\beta) = x(\tilde{k})$ and $k(\alpha) = \tilde{k}$. Then $g_{\tilde{k}}(\alpha) \neq g_{\tilde{k}}(\tilde{\gamma})$; and then $g_{\tilde{k}}(\beta) = \tilde{\gamma}$ □

Lemma 5.10. *Assume CH. Let P_{ω_1} be the forcing notion adding $\aleph_1$ many Cohen reals. Then P_{ω_1} forces $\neg\mathfrak{b}(PD^{\emptyset}(\mathrm{NS}_{\aleph_2}))$.*

(The same holds for any other CH preserving ω_1-iteration of absolute ccc forcing notions.) Note that since $\mathrm{PD}^{\emptyset}$ is monotone, $\mathfrak{b}(\mathrm{PD}^{\emptyset}(I))$ fails for all ideals I on $\aleph_2$.

Proof. Assume that $p \in P$ forces that $\underset{\sim}{\tau}$ is a winning strategy for nonempty for the game $\mathrm{PD}^{\emptyset}(I)$.

Let P_{α} be the complete subforcing of the first α Cohen reals. P_{ω_1} forces that Lemma 5.9 holds. We fix the according P_{ω_1}-names $\underset{\sim}{F}'$ and $\underset{\sim}{F}''$. Let $N \prec H(\chi)$ contain p, $\underset{\sim}{\tau}$, $\underset{\sim}{F}'$ and $\underset{\sim}{F}''$. Set $\epsilon = N \cap \omega_1$. If G_{ω_1} is P_{ω_1}-generic over V, then $G_{\epsilon} = G_{\omega_1} \cap P_{\epsilon}$ is P_{ω_1}-generic over N (and P_{ϵ}-generic over V).

So in $N_{\epsilon} = N[G_{\epsilon}] = N[G_{\omega_1}]$, we can evaluate the correct values of $\underset{\sim}{\tau}$, $\underset{\sim}{F}'$ and $\underset{\sim}{F}''$ for all valid sequences v in N_{ϵ} (i.e., the resulting values are the same as the ones calculated in V_{ω_1}).

In V_{ω_1}, pick any real $r \notin V_{\epsilon}$. Using r, we can define by induction a run b of the game such that each initial segment is in N_{ϵ}: Assume we already have the valid sequence $u \in N_{\epsilon}$. Extend u with $\underset{\sim}{F}'(u)$ if $r(n) = 0$, and to $F''(u)$ otherwise.

So $b \in V_{\omega_1}$ is a run of the game according to τ; nonempty wins the run; so there is some $\delta \in \bigcap_{n \in \omega} A(b \restriction n)$. But we can use in V_{ϵ} this δ to reconstruct by

induction the run b and therefore the real r: Assume we already know $r \restriction n$ and the corresponding the valid sequence $u = b \restriction n$. δ has to be element of exactly one of $A(F'(u))$ or $A(F''(u))$, which determines $r(n)$ and the sequence corresponding to $b \restriction (n+1)$. $\square$

On the other hand, adding Cohens, as any κ-cc forcing, preserves precipitousness (and non-precipitousness) of an ideal, cf. 4.5. So we get:

Corollary 5.11. $\mathfrak{a}(Id(I))$ *does not imply* $\mathfrak{b}(PD^{\emptyset}(\mathrm{NS}))$.

If we assume CH and an $\aleph_3$-saturated normal ideal on $\aleph_2$ saturated on $E^{\aleph_2}_{\aleph_1}$, we get the following:

Corollary 5.12. *(Saturated ideal.)* $\mathfrak{a}(BM(I))$ *does not imply* $\mathfrak{b}(PD^{\emptyset}(\mathrm{NS}))$.

Proof. Since P_{ω_1} has size $\aleph_1 < \aleph_2$, $\mathrm{cl}(I)$ remains $\aleph_3$-saturated. So in $V[G]$, we can use 3.4 to see that $\mathfrak{a}(\mathrm{BM}(\mathrm{cl}(I)))$ holds. $\square$

6. A supercompact

We have seen that $\mathfrak{b}(\mathrm{PD}(I))$ can hold for a nowhere precipitous ideal I. It seems harder to show that there could be a nowhere precipitous ideal I satisfying $\mathfrak{b}(\mathrm{PD_e}(I))$:

Theorem 6.1. *The following is consistent relative to κ supercompact: I is nowhere precipitous, i.e. $\mathfrak{a}(Id_{ne}(I))$ fails, and for every I-positive set S the dual to $I \restriction S$ can be extended to a normal ultrafilter.*

Note that this implies $\mathfrak{b}(\mathrm{PD_e}(I))$, even for the game of length κ.

Proof. Given some ordinal κ, we define the forcing $S(\kappa)$: If κ is not an inaccessible cardinal, then $S(\kappa) = \emptyset$. Otherwise, $S(\kappa)$ is the limit of the $<\kappa$-support iteration $(P_a, Q_a)_{a\in\kappa^+}$ of length κ^+ defined the following way: By induction on a, we define Q_a together with the P_a-names $B_a \subseteq \kappa$, $g_a : \kappa \to \kappa+1$ and the P_{a+1}-names $A_a \subseteq \kappa$, $f_a : \kappa \to \kappa$:

We identify the tree $T = {\kappa^+}^{<\omega}$ of finite κ^+-sequences with κ^+ such that the root is identified with 0. We can assume $a <_T b$ implies $a < b$ (as ordinals κ^+). We write $a \lhd_T b$ or $b \rhd_T a$ to denote that b is immediate T-successor of a. So for all $a \in \kappa^+$ there are κ^+ many b with $a \lhd_T b$. We also write $\mathrm{prec}(b)$ to denote the (unique) a such that $a \lhd_T b$.

Assume we already have P_a, and the P_{b+1}-name f_b for all $b < a$. Then in $V[G_{P_a}]$, we define B_a, g_a, Q_a and the Q_a-names f_a, A_a:

- If $a = 0$, we set $g_a(\alpha) = \kappa$ for all $\alpha \in \kappa$, and $B_a = \kappa$.
- Otherwise, we use some bookkeeping[8] to find a $B^0_a \subseteq A_{\mathrm{prec}(a)}$, and we set:

(8) $$B_a = B^0_a \setminus \nabla_{b<a:\,\mathrm{prec}(b)=\mathrm{prec}(a)} A_b, \text{ and we set } g_a = f_{\mathrm{prec}(a)}.$$

- A condition p of Q_a is a function $f^p : \beta^p \to \kappa$ such that $\beta^p \in \kappa$ and for all $\alpha \in \beta^p$:
  - if $\alpha \notin B_a$ or $g_a(\alpha) = 0$ then $f^p(\alpha) = 0$,
  - otherwise $f^p(\alpha) < g_a(\alpha)$.
  - Additionally, if $a = 0$ we require $f_a(\alpha) > 0$.

[8] We just need to guarantee that P_{κ^+} forces: For every $a \in T$ and every subset B of A_a there is a $b \rhd_T a$ such that $B^0_t = B$. Note that $A_b \subseteq B \subseteq A_a$.

- We define the order on Q_a by $q \leq p$ if $f^q \supseteq f^p$.
- We set f_a to be the canonical Q_a-generic, i.e., $\bigcup_{q\in G} f^q$.
- We set $A_a = \{\alpha \in \kappa : f(\alpha) > 0\}$. (So $A_0 = \kappa$.)

Note that in particular

$$\text{If } b < a \text{ and } \operatorname{prec}(b) = \operatorname{prec}(a) \text{ then } B_a \setminus A_b \text{ is nonstationary.} \tag{9}$$

(Modulo a club set, we can identify the index set of the diagonal union with κ, and then the set difference is bounded.)

So Q_a is $<\kappa$-closed. It is easy to see that P_a is $<\kappa$-directed closed: We will define P'_a by induction on $a \in \kappa^+$ and show (in the same induction) that P'_a can be interpreted to be a dense subset of P_a and is $<\kappa$-closed. A condition $p \in P'_a$ is a function (which we will call "matrix") from $u \times \beta$ to κ such that:

- $\beta \in \kappa$.
- u is a subset of a of size $<\kappa$.
- $a \triangleleft_T b$ implies $\max(1, p(a, \alpha)) > p(b, \alpha)$.
- $p(b, \alpha) > 0$ implies that $p \restriction b$ determines (as element of P_β)[9] that $\alpha \in B_b(\alpha)$.
- If $0 \in u$, then $p(0, \alpha) > 0$ for all $\alpha < \beta$.

We can interpret $p \in P'_a$ to be a condition in P_a in the obvious way; in particular we can define the order on P'_a to be the one inherited from P_a. It is easy to see that on P'_a this order is actually just extension. Obviously P'_a is $<\kappa$-closed; it remains to be shown that P'_a is dense in P_a. We do that by case distinction on $\operatorname{cf}(a)$: The case $\operatorname{cf}(a) \geq \kappa$ is trivial. Assume $a = b + 1$ and $p \in P_b$. Then by induction we know that P_a is strategically κ closed, so we can strenghten $p \restriction a$ to some $p' \in P'_a$ deciding $p(a)$ to be some f^p. We can assume that the height of p' is at least the height of f^p, and we can extend f^p up to the height of p' by adding zeros on top. So assume that a is a limit with $\operatorname{cf}(a) < \kappa$, i.e., $a = \sup b_i : i \in \lambda$ for some $b_i < a$ and $\lambda < \kappa$. Then we can define by induction on $i \in \lambda$ an increasing sequence $p'_i \in P'_i$ such that p'_i is stronger than $\bigcup_{l<i} p'_l$ as well as $p \restriction i$.

We fix a $j : V \to M$ such that

$$M \text{ is closed under } \kappa^{++}\text{-sequences.} \tag{10}$$

In particular, $\operatorname{cf}(j(\kappa)) > \kappa^+$.

We will use the reverse Easton iteration $(R_a, S(a))_{a\leq\kappa}$, for $S(a)$ defined as above. R_κ is the preparation that allows us to preserve measurability (and we will not need it for anything else); we will look at $R_\kappa * P_a$ for $a \leq \kappa^+$, and in particular at $R_{\kappa+1} = R_\kappa * P_{\kappa^+}$ (recall that $S(\kappa) = P_{\kappa^+}$). We claim that $R_{\kappa+1}$ forces what we want. We will also use $j(R_\kappa * P_a) \in M$. We get the usual properties:

- The definition of R is sufficiently absolute. In particular, we can (in M) factorize $j(R_{\kappa+1}) = R_{j(\kappa)+1}$ as $R_{\kappa+1} * R'$, where R' is the quotient forcing $R^{\kappa+1}_{j(\kappa)+1}$. Note that R' is $<\kappa^{+++}$-closed (in M and therefore in V as well).
- Assume that G is $R_{\kappa+1}$-generic over V (and M). $M[G]$ is closed (as subset of $V[G]$) under κ^+-sequences. In particular, κ^+ is the same (and also equal to 2^κ) in V, $V[G]$ and $M[G]$.
- For $p \in R_{\kappa+1}$, the domain of $j(p)$ is in $\kappa \cup \{j(\kappa)\}$, moreover $j(p) \restriction \kappa = p \restriction \kappa$ and $j(p)(j(\kappa))$ is isomorphic to $p(\kappa)$ such that $a \in \operatorname{dom}(p(\kappa))$ is mapped to $j(a)$. The image of G under j is element of $V[G]$ and subset of M of size

[9] by induction, we already know that P'_β is dense in P_β

κ^+, therefore element of $M[G]$. For $p \in G$ we can split in M the condition $j(p)$ into $p \restriction \kappa$ (which is in G anyway) and $j(p(\kappa))$. We can assume that G actually is $R_\kappa * P'_{\kappa^+}$-generic (since P'_{κ^+} is dense in P_{κ^+}). Then $j(p(\kappa))$ is a $P'_{j(\kappa^+)}$-condition. So in $M[G]$, the set $\{j(p(\kappa)) : p \in G\}$ is a directed subset of $P'_{j(\kappa^+)}$ of size κ^+, therefore the union is a $P'_{j(\kappa^+)}$-condition q_G, a matrix of height κ (which is less than $j(\kappa)$, so no contradiction to the definition of P'_a) and with domain $j''\kappa^+$ (which has size $\kappa^+ < (j(\kappa)^+)^{M[G]}$). We call this condition q_G.

- In $M[G]$, we call $q \in R'$ a G-master condition if it is stronger than q_G.
- If H contains some G-master condition and is R'-generic over $M[G]$ (and therefore $V[G]$ as well), then we can extend in $V[G][H]$ the embedding j to $V[G] \to M[G][H]$ by setting $j(\tau[G]) = j(\tau)[G][H]$. This defines in $V[G][H]$ a normal ultrafilter $U = \{A[G] : \kappa \in j(A)[G][H]\}$ over $V[G]$. Since R' is sufficiently closed, U is already element of $V[G]$.

In $V[G]$, $a \in \kappa^+$ is called a *positive index*, if

$$(\forall \zeta < j(\kappa))\,(\exists q\ G\text{-master condition})\, q \Vdash (\kappa \in j(B_a) \& j(g_a) > \zeta). \tag{11}$$

Otherwise, a is called a *null-index*. Here we interpret B_a and g_a as $R_\kappa * P_a$-names in the canonical way, so the j-images are $R_{j(\kappa)} * P_{j(a)}$-names. In particular we can assume that q is in $R_{j(\kappa)} * P_{j(a)}$ (or just in $P_{j(a)}$, since we start from $V[G]$ anyway).

If a is null and $b >_T a$, then b is null as well. (Proof: Pick $\zeta < j(\kappa)$ such that every master condition forces $j(g_a) < \zeta$ or $\kappa \notin j(B_a)$. But the empty condition forces $j(g_b)(\kappa) \leq j(g_a)\kappa$ and $j(B_b) \subseteq j(B_a)$.)

Note that 0 is a positive index.

In $V[G]$, we define the *ideal* I_0 by $A \in I_0$ iff there is an $X \subseteq \kappa^+$ of size κ consisting of null-indices such that

$$A \subseteq \nabla_{i\in X} B_i \text{ modulo a club set}^{10} \tag{12}$$

I_0 is a normal ideal on κ: Assume that $A_i \cap C_i \subseteq \nabla_{l\in X_i} B_l$ for all $i \in \kappa$. Then $(\bigcup A_i) \cup \nabla C_i \subseteq \nabla_{l\in\bigcup X_i} B_l$ modulo a club set.

By elementarity, if q is a G-master condition and if $\varphi(c, B_\alpha[G], g_\alpha[G])$ holds in $V[G]$ for some $c \in V$, then for all H containing q we get in $M[G][H]$

$$\varphi(j(c), j(B_\alpha)[G][H], j(g_\alpha)[G][H]). \tag{13}$$

We claim that in $V[G]$ the following holds:

$$\text{If } a \text{ is a positive index, then } A_a \text{ is } I_0\text{-positive.} \tag{14}$$

Note that this implies: a is a positive index iff B_a is a I_0-positive set; and I_0 is nontrivial (since 0 is a postive index).

We now prove the claim. Assume otherwise, and fix an appropriate X and a club set C, i.e.,

$$A \cap C \subseteq \nabla_{i\in X} B_i. \tag{15}$$

Since X consists of null-indices, there is for each $b \in X$ a $\zeta_b < j(\kappa)$ such that every master condition forces $\kappa \notin j(B_b)$ or $j(g_b)(\kappa) < \zeta_b$. Since $\mathrm{cf}(j(\kappa)) > \kappa$, we can find

[10] In some way, $\nabla_{i\in X} B_i$ is only defined modulo a club set, since X is not canonically isomorphic to κ (it is just a subset of κ^+ of size κ). We can from now on just fix for each such X a bijection to κ to make $\nabla_{i\in X} B_i$ well defined; still we use "subset modulo club set" in the definition on I_0.

an upper bound ξ for all ζ_b. So every master condition forces

$$\kappa \in j(B_b) \text{ implies } j(g_b)(\kappa) < \xi \text{ for all } b \in X. \tag{16}$$

Since a is a positive index, we can find a master condition q forcing $\kappa \in B_a$ and $j(g_a)(\kappa) > \xi + 1$. Without loss of generality, q is in $P_{j(a)}$. So we can extend q to q' such that

$$\kappa \in j(A) \text{ and } j(f_i)(\kappa) > \xi \tag{17}$$

Since C is club, κ is forced to be in $j(C)$. So q' forces $\kappa \in j(A_a) \cap j(C)$. According to (15), $A_a \cap C \subset \nabla_{b\in X} B_v$ holds in $V[G]$, so q' forces $\kappa \in j(A_a) \cap j(C) \subset j(\nabla_{b\in X} B_b)$. Let Z be the sequence $(B_b)_{b\in X}$. Recall that we fixed (in $V[G]$) some bijection i of κ to X, to make ∇Z well defined. So $j(\nabla Z)$ uses $j(i)$, a bijection from $j(\kappa)$ to $j(X)$; and $\kappa \in j(\nabla Z)$ means: There is an $\alpha < \kappa$ such that $\kappa \in j(Z)_{j(i)(\alpha)}$. Note that $j(Z)_{j(i)(\alpha)} = j(B_{i(\alpha)})$ and set $b = i(\alpha) \in X$. So

$$\kappa \in j(B_b) \text{ for some } b \in X \text{ (in particular, } b \text{ is null-index).} \tag{18}$$

We further extend q' to some q'' such that $q''(j(c),\kappa) = 0$ for all $c \in \kappa^+$ that are (as ordinals) bigger than α. We can do this since $j''\kappa^+ < j(\kappa^+)$. We further extend q'' to some q''' deciding the b of (18). So q''' forces

$$\kappa \in j(A_a \cap B_b) \text{ for the null-index } b. \tag{19}$$

We will get a contradiction by case distinction on the position of b relative to a in the tree T:

- $b <_T a$: This contradicts the fact that b is a nll-index and a not.
- $a \lhd_T b$: Then $g_b = f_a$, and q''' forces that $\kappa \in j(B_b)$ and $j(g_b)(\kappa) \geq j(f_b)(\kappa) > \xi > \zeta_b$, contradicting (16) and (17).
- $a \lhd_T c$ and $c <_T b$: Then c is (as an ordinal) bigger than a, and q''' forces $\kappa \notin A_c$. So $\kappa \notin B_b \subseteq A_c$.
- So a and b have to be incomparable in T, and there is some node c where a and b split. Let a' and b' the according immediate T-successors of c. So $a' \lhd_T c$, $b' \lhd_T c$, $a' \leq_T a$, $b' \leq_T b$ and $a' \neq b'$. Let $\underline{m}$ be the minimum of a', b' (as ordinals) and $\overline{m}$ the maximum. According to (9) $A_{\underline{m}} \cap B_{\overline{m}}$ is nonstationary, so $\kappa \notin j(A_{\underline{m}} \cap B_{\overline{m}})$. So (19) implies that $b' = b = \underline{m}$. Also $j(g_b)(\kappa) = j(f_c)(\kappa) \geq j(f_a)(\kappa) > \xi$ according to (17) which contradicts (16).

This proves (14).

Next we show: In $V[G]$,

$$\text{empty has a winning strategy for } \mathrm{Id}_{\mathrm{ne}}(I_0). \tag{20}$$

Assume that we have a partial run of the game of length n, corresponding to the node a in T, and empty has played X_n as last move, which is a subset of A_a. Assume that nonempty plays the I_0-positive set $B^0 \subseteq A_a$. Let $b \rhd_T a$ be such that $X_{n+1} := A_b \cap B^0$ is I_0-positive, and let X_{n+1} be empty's answer (and b be the new T-node corresponding to the new partial run). This is a winning strategy since $f_i(\alpha)$ decreases along every branch of T. It remains to be shown that we can find a $b \rhd_T a$ as above: B^0 itself is enumerated as $B^0_r cm$ by the bookkeeping at some stage $c \rhd_T a$. Recall that $B_c = B^0_c \setminus \nabla_{d<c, d\rhd_T a} A_d$. If B_c is positive, then we can set $b = c$. Otherwise, since B^0_c is positive, some $B^0_c \cap A_d$ has to be positive for some $d < c, d \rhd_T a$ (since I_0A is normal); and we can set $b = d$. This proves (20).

It remains to be shown: In $V[G]$, for every I_0-positive X there is a normal ultrafilter D_1 extending the dual of I_0 and containing X. For this, it is enough to show: If Y is I_0-positive, then there is a master condition q forcing

$$\kappa \in j(Y) \text{ and } \kappa \notin j(B_b) \text{ for all null-indices } b. \tag{21}$$

For the proof, we look at the set X of indices a such that $Y \cap A_a$ is I_0-positive. Assume $a \in X$. We will use $Y \cap A_a$ as B^0_b for some $b \rhd_T a$. We have to distinguish two cases:

- **Case 1:** There is a positice $c \rhd_T a$ such that $B_c \subseteq Y$. In particular, this will be the case if b itself is positive, i.e. if $B_b = B^0_b \setminus \nabla_{c<b, c\rhd_T a} A_c$ is I_0-positive.
- **Case 2:** There is no such c. In particular, in this case b is a null-index, so $Y \cap A_a$ is covered (modulo I_0) by $\nabla_{c<b,c\rhd_T a} A_c$. Then $c \notin X$ for any $c \geq b$ such that $c \rhd_T a$. So at most κ many immediate T-successors of a are in X; and $Y \cap A_a$ is covered (modulo I_0) by $\nabla_{c\rhd_T a, c\in X} A_c$ as well.

We claim that Case 1 has to occur for some a. Otherwise, X is a subtree of T such that every node has at most κ many successors, i.e., there are only κ many branches through X. By induction on n, X is covered (module I_0) by $\nabla_{c\in X,\ T\text{-height}(c)=n} A_c$. But for any branch b, the set $\bigcap_{n\in\omega} A_{b(n)}$ is empty (witnessed by the decreasing sequence $f_{b(n)}$), a contradiction.

So we can pick a T-minimal b such that Case 1 holds. Note that $j''\kappa^+ < \mathrm{cf}(j(\kappa))$. For every null-index c there is a witness $\xi_c < j(\kappa)$, so there is a universal bound ξ. Since b is a positive index, we can find a master condition q forcing $j(g_b)(\kappa) > \xi$ and $\kappa \in j(B_b)$. Recall that $B_b \subseteq Y \pmod{I_0}$, so q forces that $\kappa \in j(Y)$. We now extend q to q' so that it forces $\kappa \notin j(A_c)$ for all $c > a$. Then q' is as required: $\kappa \notin j(B_c)$ for any null-index c, by a similar case distinction as in the proof of (14). □

As usual, we can use a Levy collapse to reflect these properties to $\aleph_2$:

Lemma 6.2. *After collapsing κ to $\aleph_2$, we get:* $\mathrm{cl}(I)$ *is nowhere precipitous and satisfies* $\mathfrak{b}(PD_e(\mathrm{cl}(I)))$ *(even for the game of length $\aleph_1$).*

Proof. Nowhere precipitous follows from 4.5. Let S be a P-name for a $\mathrm{cl}(I)$-positive set and $p \in P$. Will show:

(22) In V there is a normal ultrafilter U and a $q \leq p$ forcing that S is $\mathrm{cl}(U)$-positive.

Then according to the usual argument, the $\mathrm{cl}(U)$-positive sets have a σ-closed dense subset, so nonempty wins $\mathrm{BM}(\mathrm{cl}(U) \restriction S)$, and — since $\mathrm{cl}(U)$ extends $\mathrm{cl}(I)$ — nonempty wins $\mathrm{PD}(I \restriction S)$ (even of length $\aleph_1$).

To prove (22), set $T = \{\alpha \in E^\kappa_{\geq\aleph_1} : p \not\Vdash \alpha \notin S\}$. T is I-positive. For each $\alpha \in T$ pick a witness $p_\alpha \leq p$. Let q, T'' be as in 6 and pick a normal ultrafilter U containing T''. We have to show that q forces S to be $\mathrm{cl}(U)$-positive. Assume otherwise, and pick $q' \leq q$ and $A \in U$ such that q' forces $A \cap S = \emptyset$. Then $q' \in Q_\alpha$ for some $\alpha < \kappa$. Pick $\beta \in T'' \cap A \setminus \alpha$. Then p_β and q' are compatible, a contradiction to $p_\beta \Vdash \beta \in S$. □

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